

# A Riemannian-Geometry Approach for Control of Robotic Systems under Constraints

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**Abstract:** A Riemannian-geometry approach for control of two-dimensional object grasping and manipulation by using a pair of multi-joint planar robot fingers is presented, together with a basic discussion on stability of position and force hybrid control of redundant robotic systems under geometric constraints. Even in the case that the shape of the object is arbitrary, it is possible to see that rolling contact constraints induce the Euler equation of motion in an implicit function form, in which constraint forces appear as wrench vectors affecting on the object. The Riemannian metric can be introduced in a natural way on a constraint submanifold induced by rolling contacts. A control signal called “blind grasping” is defined and shown to be effective in stabilization of grasping without using the details of information of object shape and parameters or external sensing. The concept of stability of the closed-loop system under constraints is renewed in order to overcome the degrees-of-freedom redundancy problem. An extension of Dirichlet-Lagrange’s stability theorem to a system of DOF-redundancy under constraints is presented by using a Morse-Lyapunov function.

**Key Words:** multi-body system, holonomic constraint, Pfaffian constraint, Riemannian submanifold, Morse-Lyapunov function.

## 1. Introduction

Continued on the previous paper to be partially presented in [1], a Riemannian geometry approach for robot control under the condition of redundancy in DOF (Degrees-of-Freedom) together with holonomic and/or nonholonomic (but Pfaffian) control is presented. Position/force hybrid control is reinterpreted in terms of “submersion” and hence the force control signal constructed in the image space of the constraint Jacobian matrix can be regarded as a lifting (or pressing) in the direction orthogonal to its kernel space. Therefore, the force control signal does not affect the geodesics on the Riemannian submanifold as a level set. By means of the submanifold distance, stability on a manifold for a redundant system under holonomic constraints is introduced. A position/force hybrid control scheme is proposed, which is shown to render the closed-loop system asymptotically stable on a manifold.

The latter part of the paper is devoted to a control problem of dexterity of a robotic hand composed of a pair of multi-joint robot fingers grasping and manipulating a two-dimensional object. Even if motion of the object/fingers system is planar, the problem was so far tackled by assuming that the object has parallel flat surfaces, that is, the object must be of rectangular. In this paper, an extension of modeling dynamics of physical interactions of rolling contact for a rigid object of two-dimensional arbitrary shape with a smooth contour is presented and geometric structures of both the kernel and image spaces of the constraint Jacobian matrix is analysed. The Euler equation of

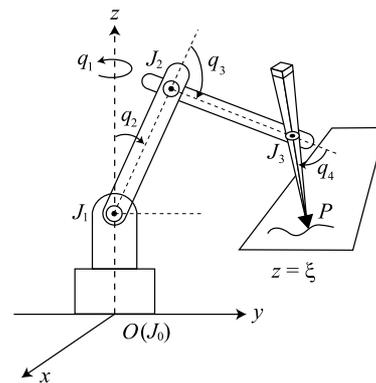


Fig. 1 A hand-writing robot with four DOFs whose endpoint  $P = (x, y)$  is constrained on a plane  $z = \xi$ .

the overall fingers/object system subject to the law-of-inertia is derived, which includes arclength parameters of the object contour curves. A couple of first-order differential equations including the curvatures of the contours should be updated in accompany with the rolling motion. It is shown that the originally nonholonomic rolling constraints are integrable in the sense of Frobenius and therefore regarded as holonomic constraints. A coordinated control signal called “blind grasping” is proposed and shown to be effective in realizing stable grasping. A sketch of the convergence proof is given on the basis of an extension of the Dirichlet-Lagrange theorem by introducing a Morse-Lyapunov function.

## 2. Constraint Submanifold, Lifting, and Riemannian Distance on a Submanifold

Consider a four-DOF robotic arm whose last link is a pencil and suppose that the endpoint of the pencil is in contact with a flat surface  $\varphi(x) = \xi$ , as shown in Fig. 1. The Lagrange equation of motion of the system is written by the form

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$$G(q)\ddot{q} + \left\{ \frac{1}{2}\dot{G} + S \right\} \dot{q} + g(q) = -\lambda \frac{\partial \varphi(\mathbf{x}(q))}{\partial q} \quad (1)$$

where  $G(q)$  denotes the inertia matrix of the arm,  $S$  a skew-symmetric matrix, and  $\lambda$  the Lagrange multiplier. In this paper, the set of all possible postures of the robot is regarded as a Riemannian manifold denoted by  $\{M, p, g_{ij}\}$ , where  $G(q) = (g_{ij})$ . Here,  $\partial\varphi/\partial q$  can be decomposed into  $\partial\varphi(q)/\partial q = J^T(q)\partial\varphi/\partial\mathbf{x}$ , where  $J(q) = \partial\mathbf{x}/\partial q^T$ . On the constraint Riemannian manifold  $F_\xi = \{p | p \in M \text{ and } \varphi(\mathbf{x}(p)) = \xi\}$ , let us consider a smooth curve  $c(t) : I[a, b] \rightarrow F_\xi$  that connects the given two points  $c(a) = p$  and  $c(b) = p'$  where  $p$  and  $p'$  belong to  $F_\xi$ . The length of such a curve constrained to  $F_\xi$  is defined as

$$L(c) = \int_a^b \sqrt{g_{ij}(c(t))\dot{c}^i(t)\dot{c}^j(t)} dt \quad (2)$$

and consider the minimization

$$d(p, p') = \inf_{c \in F_\xi} L(c) \quad (3)$$

that should be called the distance between  $p$  and  $p'$  on the constraint submanifold. Then, the minimizing curve called the geodesic denoted identically by  $q(t) (= c(t))$  must satisfy the Euler equation

$$\ddot{q}^k(t) + \Gamma_{ij}^k \dot{q}^i(t)\dot{q}^j(t) = -\lambda(t) \cdot (\text{grad } \varphi(\mathbf{x}(t)))^k \quad (4)$$

together with the constraint condition  $\varphi(\mathbf{x}(t)) = \xi$ , where

$$\text{grad } \varphi(\mathbf{x}(t)) = G^{-1}(q(t))J^T(q) \frac{\partial \varphi}{\partial \mathbf{x}} \quad (5)$$

From (4) and the inner product of (4) and  $w = J^T \partial\varphi/\partial\mathbf{x}$  it follows that

$$\sum_k (w^k \ddot{q}^k + w^k \Gamma_{ij}^k \dot{q}^i \dot{q}^j) = -\lambda \sum_k w^k g^{kj} w^j \quad (6)$$

From the Riemannian metric introduced over  $u, v \in TF_\xi$ , tangent space of  $F_\xi$ , by  $\langle u, v \rangle = g_{ij}u^i v^j$  (summation over  $i, j$  is omitted), we see (see [1]) that  $\langle w, \dot{q} \rangle = 0$ , which implies

$$\sum_{k=1}^4 \left( w^k \ddot{q}^k + \left( \frac{d}{dt} w^k \right) \dot{q}^k \right) = 0 \quad (7)$$

Substituting this into (6) yields

$$\lambda(t) = \frac{1}{w^T G^{-1} w} \left\{ \sum_k \left( \dot{w}^k \dot{q}^k - \sum_{i,j} w^k \Gamma_{ij}^k \dot{q}^i \dot{q}^j \right) \right\} \quad (8)$$

From the Riemannian geometry, the constraint force  $\lambda(t)$  with  $\text{grad } \{\varphi(\mathbf{x}(q))\}$  stands for a component of the image space of  $w (= J^T(q)\partial\varphi/\partial\mathbf{x})$  that is orthogonal to the kernel  $TF_\xi$  of  $w$ . In other words this component is cancelled out by the image space component of the left hand side of (6). From the physical point of view,  $\lambda(t)$  should be regarded as a magnitude of the constraint force that presses the surface  $\varphi(\mathbf{x}(q)) = \xi$  in its normal direction. In order to compromise the mathematical argument with such physical reality, let us suppose that the actuators can supply the torque signal

$$u = \lambda_d \cdot J^T(q) \frac{\partial \varphi}{\partial \mathbf{x}} + g(q) \quad (9)$$

Then, by substituting this into (1) we obtain the Lagrange equation of motion under the constraint  $\varphi(\mathbf{x}(q)) = \xi$ :

$$\begin{aligned} G(q)\ddot{q} + \left\{ \frac{1}{2}\dot{G} + S \right\} \dot{q} &= -(\lambda - \lambda_d)J^T(q) \frac{\partial \varphi}{\partial \mathbf{x}} \\ &= -\Delta\lambda J^T(q) \frac{\partial \varphi}{\partial \mathbf{x}} \end{aligned} \quad (10)$$

where  $\Delta\lambda = \lambda - \lambda_d$ . It should be noted that introduction of the first term of control signal of (9) does not affect the solution orbit on the constraint manifold and further it keeps the constraint condition during motion by rendering  $\lambda(t) (= \lambda_d + \Delta\lambda(t))$  positive. In a mathematical sense, exertion of the joint torque  $\lambda_d J^T(\partial\varphi/\mathbf{x})$  plays a role of ‘‘lifting’’ of (or ‘‘pressing’’ to) the image space of a gradient of the constraint equation. Further, note that (10) is of an implicit function form with the Lagrange multiplier  $\Delta\lambda$ . In order to assure the argument in treatment of the geodesics through this implicit form, we will also show an explicit form of the Lagrange equation expressed on the orthogonally projected space (kernel space) by introducing the orthogonal coordinate transformation

$$\dot{q} = (P, w\|w\|^{-1}) \begin{pmatrix} \dot{\eta} \\ \dot{z} \end{pmatrix} = Q\dot{q} \quad (11)$$

where  $P$  is a  $4 \times 3$  matrix whose column vectors with the unit norm are orthogonal to  $w$  and  $\dot{\eta}$  denotes a  $3 \times 1$  matrix (3-dim. vector) and  $\dot{z}$  a scalar. Since  $Q$  is an orthogonal matrix,  $Q^{-1} = Q^T$ . Hence, if  $\dot{q} \in \ker(w) (= TF_\xi)$  then  $\dot{z} = 0$ . Restriction of (10) to the kernel space of  $w$  can be obtained by multiplying (10) by  $P^T$  from the left in such a way that

$$P^T G(q) \frac{d}{dt} (P\dot{\eta}) + P^T \left\{ \frac{1}{2}\dot{G} + S \right\} P\dot{\eta} = 0 \quad (12)$$

which is reduced to the Euler equation in  $\dot{\eta}$ :

$$\bar{G}(q)\ddot{\eta} + \left\{ \frac{1}{2}\dot{\bar{G}} + \bar{S} \right\} \dot{\eta} = 0 \quad (13)$$

or equivalently

$$\ddot{\eta}^k + \bar{\Gamma}_{ij}^k \dot{\eta}^i \dot{\eta}^j = 0, \quad k = 1, \dots, n-1 (= 3) \quad (14)$$

where  $\bar{G}(q) = P^T G(q) P$ ,  $\bar{\Gamma}_{ij}^k$  the Christoffel's symbol for  $(\bar{g}_{ij}) = \bar{G}$ , and

$$\bar{S} = P^T S P - \frac{1}{2} \dot{P}^T G P + \frac{1}{2} P^T G \dot{P} \quad (15)$$

which is skew-symmetric, too [1]. Note that the transformation  $Q$  is isometric and equation (14) stands for the geodesic equation on the constraint Riemannian submanifold.

### 3. Position/Force Hybrid Control for Redundant Systems

Let us now consider the position/force hybrid control for the redundant robot with four joints constrained on a plane  $\varphi(\mathbf{x}) = \xi$  when  $\varphi(\mathbf{x}) = z$  (see Fig. 1). Suppose that a target endpoint position  $\mathbf{x} = \mathbf{x}_d = (x_d, y_d, z_d = \xi)$  is given together with the target pressing force  $\lambda = \lambda_d$ . Similarly to the control signal proposed by McClamoro and Wang [2],[3], we consider the control signal

$$u = g(q) - C\dot{q} + \lambda_d J^T(q) \frac{\partial \varphi}{\partial \mathbf{x}} - J^T(q) (\zeta \sqrt{k} \dot{\mathbf{x}} + k \Delta \mathbf{x}) \quad (16)$$

where  $\varphi(\mathbf{x}) = z$ ,  $\dot{\mathbf{x}} = (\dot{x}, \dot{y}, 0)^T$ , and  $\Delta \mathbf{x} = (x - x_d, y - y_d, 0)^T$ . Substituting this into (1) yields

$$\begin{aligned} G(q)\ddot{q} + \left\{ \frac{1}{2}\dot{G} + S + C \right\} \dot{q} + J^T(q) \left\{ \zeta \sqrt{k} \dot{\mathbf{x}} + k \Delta \mathbf{x} \right\} \\ = -\Delta\lambda \frac{\partial z}{\partial q} \end{aligned} \quad (17)$$

The inner product of (17) and  $\dot{q}$  under the constraint yields

$$\frac{d}{dt}E(q, \dot{q}) = -\dot{q}^T C \dot{q} - \zeta \sqrt{k} \|\dot{\mathbf{x}}\|^2 \quad (18)$$

where

$$E(q, \dot{q}) = \frac{1}{2} \dot{q}^T G(q) \dot{q} + \frac{k}{2} \|\Delta \mathbf{x}\|^2 \quad (19)$$

Unfortunately this quantity  $E(q, \dot{q})$  is not positive definite in the tangent bundle  $M \times TF_\xi$ . Nevertheless, it is possible to see that magnitudes of  $\dot{q}$  and  $\Delta \mathbf{x}$  remain small if, at initial time  $t = 0$ ,  $\|\dot{q}(0)\|$  and  $\|\Delta \mathbf{x}(0)\|$  are appropriately small. Let us now introduce the equilibrium manifold as

$$EM_1 = \{q \mid z(q) = \xi, x(q) = x_d, \text{ and } y(q) = y_d\} \quad (20)$$

which is of one-dimension. To discuss stability of an equilibrium point  $q^*$  belonging to  $EM_1$  or convergence of an orbit  $q(t)$  of (17) to some other point on  $EM_1$  with increasing  $t$ , we need to introduce the concept of neighborhoods in the constraint manifold by defining a Riemannian ball around  $q^* \in EM_1$

$$B(q^*, r_0) = \{q \mid d(q, q^*) < r_0, q \in F_\xi\} \quad (21)$$

with some positive number  $r_0 > 0$  such that the Jacobian matrix  $J(q) (= \partial(x, y)^T / \partial q^T)$  is nondegenerate and there exist positive numbers  $\sigma_m$  and  $\sigma_M$  satisfying

$$\sigma_M I_2 \geq J(q) J^T(q) \geq J(q) P_\varphi J^T(q) \geq \sigma_m I_2 \quad (22)$$

where  $P_\varphi = I_4 - w w^T / \|w\|^2$ . Following the definition of stability on a manifold previously defined in the case that the robot endpoint is free to move (see [1]), we define:

**Definition a)** If, for any  $\varepsilon > 0$ , there exist  $\delta(\varepsilon) > 0$  and  $r_1 > 0$  (that is independent of  $\varepsilon$  but may be less than  $r_0$ ) such that  $q(t)$  of a solution of (17) starting from  $(q(0), \dot{q}(0) = 0)$  in  $B(q^*, r_1)$  with  $\|\mathbf{x}(q(0)) - \mathbf{x}_d\| < \delta$  remains inside  $B(q^*, r_0)$ , its endpoint satisfies  $\|\mathbf{x}(t) - \mathbf{x}_d\| < \varepsilon$ , and approaches asymptotically to the EP-manifold  $EM_1$  with  $\dot{q}(t) \rightarrow 0$  and  $\Delta \lambda(t) \rightarrow 0$ , then the equilibrium point  $q^*$  is said to be stable on EP-manifold.

It should be remarked that the quantities  $\varepsilon$  and  $\delta(\varepsilon)$  are taken on the basis of physical unit [m] in  $E^2$  or  $E^3$  but  $r_0$  and  $r_1$  are based on the Riemannian metric originally introduced for measuring the distance  $d(q, \bar{q})$  between two postures  $q$  and  $\bar{q}$  of the robot. The condition that the robot posture  $q(t)$  should remain in a Riemannian ball during its motion is indispensable in order to avoid possibility of incurring self-motion that may arise in the case of redundancy in the system's degrees-of-freedom.

Another definition of a more severe stability motion is given as follows:

**Definition b)** If for any  $\varepsilon > 0$  there exists a number  $\delta(\varepsilon) > 0$  such that any trajectory of (17) starting from an arbitrary initial position  $q(0)$  inside  $B(q^*, \delta(\varepsilon))$  with  $\dot{q}(0) = 0$  remains inside  $B(q^*, \varepsilon)$  for any  $t > 0$  and further approaches asymptotically to some posture  $q^\infty$  on  $EM_1$  together with  $\dot{q}(t) \rightarrow 0$  and  $\Delta \lambda(t) \rightarrow 0$ , then the posture  $q^*$  on  $EM_1$  is said to be stable on a submanifold (see Fig. 2).

In this severe definition of stability of motion, numbers  $\varepsilon$  and  $\delta(\varepsilon)$  are based on the Riemannian metric originally introduced on the Riemannian manifold  $\{M, p, g_{ij}\}$ . Since the function  $\mathbf{x}(q)$  is of  $C^\infty$ -class as a mapping  $\mathbf{x} : M \rightarrow E^2$ , the stability of b) implies that of a).

We are now in a position to show that, in most ordinary cases of robotic systems, any equilibrium position  $q^*$  having a Riemannian ball  $B(q^*, r_0)$  that satisfies (22) is stable in either meaning a) or b). To do this, let us introduce a scalar quantity

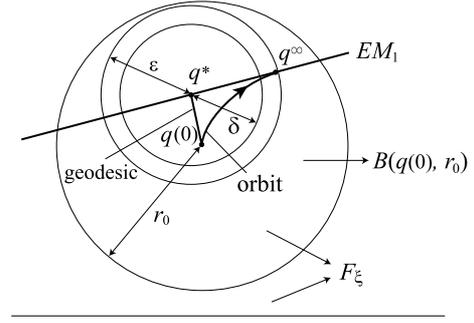


Fig. 2 Definition of the Riemannian ball  $B(q^*, \delta)$ . Any orbit starting from  $B(q^*, \delta)$  converges to  $EM_1 \cap B(q^*, \varepsilon)$ .

$$V(q, \dot{q}) = \frac{1}{2} \zeta \sqrt{k} \|\Delta \mathbf{x}\|^2 + \dot{q}^T G J^+ \Delta \mathbf{x} \quad (23)$$

and consider

$$W_\alpha(q, \dot{q}) = E(q, \dot{q}) + \alpha V(q, \dot{q}) \quad (24)$$

where  $\alpha > 0$  is a parameter less than or equal to 1 and  $J^+(q) = P_\varphi(q) J^T(q) \{J(q) P_\varphi(q) J^T(q)\}^{-1}$  that is nothing else but the pseudo-inverse of  $J(q) P_\varphi(q)$ . Then, the derivative of  $W_\alpha$  in  $t$  along the solution trajectory to (17) is given by

$$\begin{aligned} \dot{W}_\alpha = & -c \|\dot{q}\|^2 - \zeta \sqrt{k} \|\dot{\mathbf{x}}\|^2 - \alpha k \|\Delta \mathbf{x}\|^2 \\ & + \alpha \dot{q}^T G J^+ \dot{\mathbf{x}} + \alpha h(\dot{q}, G) \Delta \mathbf{x} \end{aligned} \quad (25)$$

where we set  $C = c I_4$  with a constant  $c > 0$  to simplify the argument and define

$$h(\dot{q}, G) = \dot{q}^T \left\{ \left( \frac{1}{2} \dot{G} P_\varphi + S P_\varphi \right) J^+ + G J^+ \right\} \quad (26)$$

where we denote  $dJ^+(q)/dt$  by  $\dot{J}^+$  for simplicity. Note that the  $1 \times 2$ -vector  $h$  is quadratic in components of  $\dot{q}$  and each coefficient of  $\dot{q}_i \dot{q}_j$  (for  $i, j = 1, \dots, 4$ ) is at most of order of the maximum eigenvalue of  $G(q)$  denoted by  $g_M$ . Hence, it follows from (25) that

$$|h(\dot{q}, G) \Delta \mathbf{x}| \leq g_M l_0 \sigma_m^{-1} \|\dot{q}\|^2 \|\Delta \mathbf{x}\| \quad (27)$$

where  $l_0$  signifies a constant that is of order of the maximum link length (because the Jacobian matrix  $J(q)$  is homogeneously related to each of the three link lengths of the robot shown in Fig. 1). Further, note that

$$\begin{aligned} \dot{q}^T G J^+ \dot{\mathbf{x}} & \leq \frac{\gamma}{2} \dot{q}^T G G \dot{q} + \frac{1}{2\gamma} \dot{\mathbf{x}}^T (J^+)^T J^+ \dot{\mathbf{x}} \\ & \leq \frac{\gamma g_M}{2} \dot{q}^T G \dot{q} + \frac{1}{2\gamma \sigma_m} \|\dot{\mathbf{x}}\|^2 \end{aligned} \quad (28)$$

where  $\gamma$  is an arbitrary positive parameter. Let us now set the parameter  $\gamma$  as

$$1/\gamma = 2\sigma_m \zeta \sqrt{k} \quad (29)$$

and choose the damping coefficient  $c > 0$  so that it satisfies

$$c \geq \max\{3g_M, 2\gamma g_M^2\} \quad (30)$$

Then, substituting (27) and (28) into (25) by referring to (29) and (30) yields

$$\dot{W}_\alpha \leq -\frac{3c}{4} \|\dot{q}\|^2 - \alpha k \|\Delta \mathbf{x}\|^2 + \frac{\alpha g_M l_0}{\sigma_m} \|\Delta \mathbf{x}\| \|\dot{q}\|^2 \quad (31)$$

As for ordinary sizes of a hand-writing robot shown in Fig. 1,  $g_M$  and  $l_0$  have physical scales such as

$$g_M \leq 0.001 \text{ [kgm}^2\text{]}, \quad l_0 \leq 0.25 \text{ [m]} \quad (32)$$

and  $k$  with the unit [N/m] is chosen between  $k = 5.0 \sim 50.0$  [N/m] and  $\zeta$  is set around  $\zeta = 1.0$ . By this choice for  $k$  and  $\zeta$ , we have  $k > 2\zeta\sqrt{k}$ . Similarly to (28), note that

$$\dot{q}^T G J^+ \Delta \mathbf{x} \leq \frac{\gamma g_M}{2} \dot{q}^T G \dot{q} + \frac{1}{2\gamma\sigma_m} \|\Delta \mathbf{x}\|^2 \quad (33)$$

Hence, it follows from (30) that

$$\begin{aligned} W_\alpha &= \frac{1}{2} \dot{q}^T G \dot{q} + \frac{k}{2} \|\Delta \mathbf{x}\|^2 + \frac{\alpha\zeta\sqrt{k}}{2} \|\Delta \mathbf{x}\|^2 + \alpha \dot{q}^T G J^+ \Delta \mathbf{x} \\ &\leq \frac{1 + \alpha g_M \gamma}{2} \dot{q}^T G \dot{q} + \frac{1}{2} \left( k + \alpha\zeta\sqrt{k} + \frac{\alpha}{\gamma\sigma_m} \right) \|\Delta \mathbf{x}\|^2 \\ &\leq \frac{c}{2} \|\dot{q}\|^2 + k \|\Delta \mathbf{x}\|^2 \end{aligned} \quad (34)$$

for any parameter value for  $0 < \alpha \leq 1$ . If  $\sigma_m$  is not so small that it satisfies

$$g_M / \sigma_m \zeta \sqrt{k} = 2\gamma g_M \geq 1 \quad (35)$$

then we set  $\alpha = 1.0$  as the maximum value. Otherwise, choose

$$\alpha = \frac{\sigma_m \zeta \sqrt{k}}{g_M} \left( = \frac{1}{2\gamma g_M} \right) \quad (36)$$

Then, from a similar argument in derivation of (34) it follows that

$$\begin{aligned} W_\alpha &\geq \frac{1 - \alpha g_M \gamma}{2} \dot{q}^T G \dot{q} + \frac{k + \alpha\zeta\sqrt{k} - \alpha/\gamma\sigma_m}{2} \|\Delta \mathbf{x}\|^2 \\ &\geq \frac{1}{4} \dot{q}^T G \dot{q} + \frac{k}{4} \|\Delta \mathbf{x}\|^2 \geq \frac{1}{2} E(q, \dot{q}) \end{aligned} \quad (37)$$

That is,  $W_\alpha(q, \dot{q})$  is positive definite in  $\dot{q}$  and  $\Delta \mathbf{x}$ .

Consider now a solution trajectory  $(q(t), \dot{q}(t))$  to (17) starting from  $(q(0), \dot{q}(0) = 0)$  for  $q(0) \in B(q^*, r_1)$  with some  $r_1 (\leq r_0)$  with  $\|\mathbf{x}(q(0)) - \mathbf{x}_d\| \leq \delta$  for some  $\delta > 0$ . For the time being, assume that the solution orbit  $q(t)$  remains inside  $B(q^*, r_0)$ . During that period it is important to note that

$$\|\mathbf{x}(q(t)) - \mathbf{x}_d\| \leq \delta \quad (38)$$

because  $E(q(t), \dot{q}(t))$  is non-increasing with increase of  $t$  according to (18). Therefore, if we choose

$$\delta(\varepsilon) = \min \left\{ \varepsilon, \frac{c\sigma_m}{4g_M l_0}, \frac{\alpha r_0}{4\sqrt{3k}} \right\} \quad (39)$$

then  $\|\Delta \mathbf{x}(0)\| < \delta(\varepsilon)$  implies that

$$\frac{g_M l_0}{\sigma_m} \|\Delta \mathbf{x}(t)\| \leq \frac{g_M l_0}{\sigma_m} \|\Delta \mathbf{x}(0)\| \leq \frac{c}{4} \quad (40)$$

from which (31) is reduced to

$$\dot{W}_\alpha \leq -\frac{c}{2} \|\dot{q}\|^2 - \alpha k \|\Delta \mathbf{x}\|^2 \quad (41)$$

The reason of existence of the last member  $\alpha r_0 / 4\sqrt{3k}$  in  $\{\}$  of (39) will be explained later. This inequality (39) together with (34) means

$$\dot{W}_\alpha \leq -\alpha W_\alpha \quad (42)$$

which shows

$$W_\alpha(q(t), \dot{q}(t)) \leq W_\alpha(q(0), \dot{q}(0)) e^{-\alpha t} = \frac{3k}{4} \|\Delta \mathbf{x}(0)\|^2 e^{-\alpha t} \quad (43)$$

Along the solution trajectory of (17), we now conclude that

$$\begin{aligned} \int_0^t \sqrt{\langle \dot{q}(\tau), \dot{q}(\tau) \rangle} d\tau &= \int_0^t \sqrt{\dot{q}^T(\tau) G(q(\tau)) \dot{q}(\tau)} d\tau \\ &\leq \int_0^t \sqrt{2E(q(\tau), \dot{q}(\tau))} d\tau \leq \int_0^t \sqrt{4W_\alpha(\tau)} d\tau \\ &\leq \sqrt{3k} \|\Delta \mathbf{x}(0)\| \int_0^t e^{-(\alpha/2)\tau} d\tau \leq \frac{2\sqrt{3k}}{\alpha} \|\Delta \mathbf{x}(0)\| \\ &\leq \frac{2\sqrt{3k}}{\alpha} \delta \end{aligned} \quad (44)$$

Since the Riemannian distance of any geodesic curve  $c(t) : I = [a, b] \rightarrow F_\xi$  that connects the postures  $c(a) = q(0)$  and  $c(b) = q(T)$  is invariant under the change of length parameter such as  $s = \beta t + \eta$  with any constant  $\beta > 0$  and  $\eta$ , we see that from (3) and (44)

$$d(q(0), q(t)) = \inf_{c \in F_\xi} L(c) = \int_0^t \sqrt{g_{ij} \dot{q}^i \dot{q}^j} d\tau \leq \frac{2\sqrt{3k}}{\alpha} \delta \quad (45)$$

Thus, once we choose  $\delta(\varepsilon)$  for any arbitrary given  $\varepsilon > 0$  so that it satisfies (39) and set  $r_1 = r_0/2$ , we obtain

$$\begin{aligned} d(q(t), q^*) &\leq d(q(t), q(0)) + d(q(0), q^*) \\ &< \frac{2\sqrt{3k}}{\alpha} \delta(\varepsilon) + r_1 \leq \frac{r_0}{2} + \frac{r_0}{2} = r_0 \end{aligned} \quad (46)$$

provided that  $q(0) \in B(q^*, r_1)$  and  $\|\Delta \mathbf{x}(0)\| < \delta(\varepsilon)$ . Further, asymptotic convergences of  $\dot{q}(t) \rightarrow 0$  and  $\mathbf{x}(t) \rightarrow \mathbf{x}_d$  as  $t \rightarrow \infty$  follow from the exponential convergence of  $W_\alpha(q(t), \dot{q}(t))$  and  $E(q(t), \dot{q}(t))$  to zero as  $t \rightarrow \infty$ . All these facts imply that  $q(t)$  approaches asymptotically some point  $q^\infty \in EM_1 \cap B(q^*, r_0)$ . Finally, note that asymptotic convergence of the solution trajectory to some  $q^\infty$  on the equilibrium manifold implies also the asymptotic convergence of constraint force  $\lambda(t)$  to  $\lambda_d$  as  $t \rightarrow \infty$  because  $\Delta \lambda = \lambda - \lambda_d$  is expressed as

$$\begin{aligned} \Delta \lambda(t) &= \frac{1}{w^T G^{-1} w} \left\{ \sum_k \left( \dot{w} \dot{q}^k - \sum_{i,j} w^k \Gamma_{ij}^k \dot{q}^i \dot{q}^j \right) \right. \\ &\quad \left. - w^T J^T \left( \zeta \sqrt{k} \dot{\mathbf{x}} + k \Delta \mathbf{x} \right) \right\} \end{aligned} \quad (47)$$

and this right hand side converges to zero as  $t \rightarrow \infty$ .

The stability notion of a Riemannian ball in a neighborhood of a reference equilibrium state  $q^*$  on  $EM_1$  is extended to cope with the case that the initial velocity vector  $\dot{q}(0)$  is not zero. To do this, we define an extended Riemannian ball in the tangent bundle  $M \times TF_\xi$  around  $(q^*, \dot{q} = 0)$  in such a way that

$$\begin{aligned} B\{(q^*, 0); (r_0, r_K)\} \\ = \left\{ (q, \dot{q}) : d(q, q^*) < r_0, \sqrt{(1/2) \dot{q}^T \bar{G}(q) \dot{q}} < r_K \right\} \end{aligned} \quad (48)$$

where  $d(q, q^*)$  denotes the distance between  $q$  and  $q^*$  restricted to the submanifold  $F_\xi$  and  $\bar{G}$  is defined below (14).

**Definition** (Asymptotic Stability on a Submanifold)

If for any  $\varepsilon > 0$  there exist numbers  $\delta(\varepsilon)$  and  $\delta_K(\varepsilon)$  such that any solution trajectory of (17) starting from an arbitrary initial position and velocity inside  $B\{(q^*, 0); (\delta(\varepsilon), \delta_K(\varepsilon))\}$  remains in  $B\{(q^*, 0); (\varepsilon, r_K)\}$  and further converges asymptotically to some equilibrium point  $q^\infty \in EM_1$  with still state, then the reference

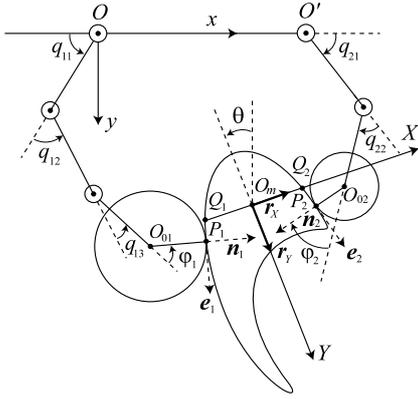


Fig. 3 Two-dimensional object grasping by a pair of multi-joint robot fingers.

point with its posture on  $EM_1$  is said to be asymptotically stable on a constraint submanifold.

It should be remarked that Bloch et al.[4] introduces originally the concept of stabilization for a class of nonholonomic dynamic systems based upon a certain configuration space. The redefinition of stability concepts introduced above is free from any choice of configuration spaces (local coordinates) and assumptions on an invertibility condition (that is almost equivalent to nonlinear control based on compensation for nonlinear inertia-originated terms). Liu and Li [5] also gave a geometric approach to modeling of constrained mechanical systems based upon orthogonal projection maps without deriving a compact explicit form of the Euler equation like (14) with a reduced dimension due to constraints. Therefore, the proposed control scheme was developed on the basis of compensation for the inertia-originated nonlinear terms (that is almost equivalent to the computed torque method). A naive idea of stability on a manifold by using different metrics for the constrained submanifold and its tangent space was first presented in [6] and used in stabilization control of robotic systems with DOF-redundancy [7],[8].

Finally it should be remarked that the artificial potential in task space  $(k/2)\|\Delta x(q)\|^2$  can be regarded as a Morse function with the constraint  $\varphi(x) = \xi$  on a differentiable manifold  $(M, p)$ . The stability result is also regarded as an extension of the Dirichlet-Lagrange theorem of stability of motion to the case of robot motion under redundancy in degrees-of-freedom and a holonomic constraint.

#### 4. 2-dimensional Stable Grasp of a Rigid Object with Arbitrary Shape

Consider a control problem for stable grasping of a 2-D rigid object by a pair of planar multi-joint robot fingers with hemispherical finger-tips as shown in Fig. 3. In this figure, the two robots are installed on the horizontal  $xy$ -plane  $E^2$ . We denote the object mass center by  $O_m$  with the coordinates  $(x_m, y_m)$  expressed in the inertial frame. On the other hand, we express a local coordinate system fixed at the object by  $O_m$ - $XY$  together with unit vectors  $r_X$  and  $r_Y$  along the  $X$ -axis and  $Y$ -axis respectively (see Fig. 4). The left-hand side surface of the object is expressed by a curve  $c(s)$  with local coordinates  $(X(s), Y(s))$  in terms of length parameter  $s$  as shown in Fig. 4.

First, suppose that at the left-hand contact point  $P_1$  the finger-tip of the left finger is contacting with the object. Denote the

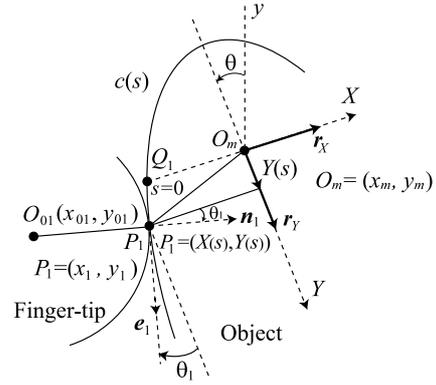


Fig. 4 Definitions of physical symbols in relation to rolling contact.  $\theta_1$  denotes  $\theta_1(s)$  appearing in (49).

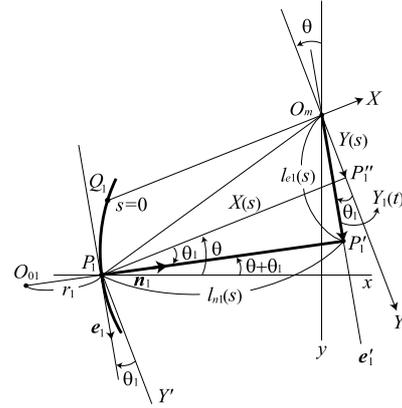


Fig. 5 Physical and geometrical meanings of key variables  $l_{n1}(s)$  and  $l_{e1}(s)$  defined in relation to the object contour  $c(s)$  expressed in terms of length parameter  $s$  of the curve.

unit normal at  $P_1$  by  $n_1$  and the unit tangent vector by  $e_1$ . Note that  $n_1$  is normal to both the object surface and finger-end sphere at  $P_1$  and  $e_1$  is tangent to them at  $P_1$ , too. If we denote position  $P_1$  by local coordinates  $(X(s), Y(s))$  fixed at the object (see Fig. 4), then the angle from the  $X$ -axis to the unit normal  $n_1$  is assumed to be determined by a function on the curve:

$$\theta_1(s) = \arctan(Y'(s)/X'(s)) \quad (49)$$

where  $X'(s) = dX(s)/ds$  and  $Y' = dY(s)/ds$ . Then,

$$\overline{P_1 P'_1} = l_{n1}(s) = -X(s) \cos \theta_1(s) + Y(s) \sin \theta_1(s) \quad (50)$$

which is dependent only on  $s$  and therefore a shape-function of the object. On the other hand, this length can be expressed by using the inertial frame coordinates as (see Fig. 5)

$$\overline{P_1 P'_1} = (x_m - x_{01}) \cos(\theta + \theta_1(s)) - (y_m - y_{01}) \sin(\theta + \theta_1(s)) - r_1 \quad (51)$$

Hence, the left-hand contact constraint can be expressed by the holonomic constraint

$$Q_1 = -(x_m - x_{01}) \cos(\theta + \theta_1) + (y_m - y_{01}) \sin(\theta + \theta_1) + (r_1 + l_{n1}(s)) = 0 \quad (52)$$

where  $l_{n1}(s)$  denotes the right hand side of (50) and  $\theta_1 = \theta_1(s)$  for abbreviation.

Next, we note that the length  $\overline{O_m P'_1}$  can be also regarded as a shape-function of the object as in the following:



$$G(X)\ddot{X} + \left\{ \frac{1}{2}\dot{G} + S \right\} \dot{X} + f_1 \frac{\partial Q_1}{\partial X} + f_2 \frac{\partial Q_2}{\partial X} + \lambda_1 \frac{\partial R_1}{\partial X} + \lambda_2 \frac{\partial R_2}{\partial X} = \mathbf{u} \quad (75)$$

According to change of the contact points  $P_1$  and  $P_2$  rolling on their corresponding finger-end spheres and object side surfaces, the arclength parameters  $s_1$  and  $s_2$  should be updated in the following way

$$\frac{ds_i}{dt} = \frac{-(-1)^i r_i}{1 + r_i \kappa_i(s_i)} (\dot{p}_i - \dot{\theta}), \quad i = 1, 2 \quad (76)$$

where  $\kappa_i(s_i)$  denote curvatures of the object contours at contacts. Derivation of (76) is discussed in Appendix.

## 5. Lifting in Horizontal Space and Force/Torque Balance

The Euler equation of (75) can be expressed in a similar form to (4) by introducing the constraint vector  $\Phi$  and the vector of Lagrange multipliers  $\lambda$  as

$$\Phi = (Q_1, R_1, Q_2, R_2), \quad \lambda = (f_1, \lambda_1, f_2, \lambda_2)^T \quad (77)$$

Then, (75) can be written in the form

$$G(X)\ddot{X} + \left\{ \frac{1}{2}\dot{G} + S \right\} \dot{X} + \frac{\partial \Phi}{\partial X} \lambda = \mathbf{u} \quad (78)$$

which must be valid for the constraint  $\Phi = (0, 0, 0, 0)$ . We denote the  $(n-4)$ -dimensional kernel space of  $\partial\Phi/\partial X$  by  $V_X$  and its 4-dimensional orthogonal complement as the image space of  $\partial\Phi/\partial X$  by  $H_X$ .

First, in order to find an adequate lifting that belongs to the image space  $H_X$  and realizes the force/torque balance (see (72)) in the sense that

$$f_{1d}\mathbf{w}_1 + f_{2d}\mathbf{w}_2 + \lambda_{1d}\mathbf{w}_3 + \lambda_{2d}\mathbf{w}_4 = 0 \quad (79)$$

we remark that

$$\begin{pmatrix} x_m - x_{01} \\ y_m - y_{01} \end{pmatrix} = R_{\theta+\theta_1} \begin{pmatrix} r_1 + l_{n1}(s_1) \\ -Y_1 \end{pmatrix} \quad (80)$$

$$-\begin{pmatrix} x_m - x_{02} \\ y_m - y_{02} \end{pmatrix} = R_{\theta+\theta_2} \begin{pmatrix} l_{n2}(s_2) + r_2 \\ Y_2 \end{pmatrix} \quad (81)$$

where

$$R_{\theta+\theta_i} = (-\mathbf{n}_1(\theta), \mathbf{e}_1(\theta)) = \begin{pmatrix} \cos(\theta + \theta_1) & \sin(\theta + \theta_1) \\ -\sin(\theta + \theta_1) & \cos(\theta + \theta_1) \end{pmatrix} \quad (82)$$

We define  $R_{\theta+\theta_2}$  and  $R_\theta$  similarly. Then, it follows from (80) and (81) that

$$\begin{aligned} & -\begin{pmatrix} x_{01} - x_{02} \\ y_{01} - y_{02} \end{pmatrix} \\ & = R_\theta \left\{ R_{\theta_1} \begin{pmatrix} r_1 + l_{n1}(s_1) \\ -Y_1 \end{pmatrix} + R_{\theta_2} \begin{pmatrix} r_2 + l_{n2}(s_2) \\ Y_2 \end{pmatrix} \right\} \\ & = R_\theta \begin{pmatrix} l(s, Y_1, Y_2) \\ -d(s, Y_1, Y_2) \end{pmatrix} \end{aligned} \quad (83)$$

where  $\mathbf{s} = (s_1, s_2)$  and

$$\begin{cases} l(s, Y_1, Y_2) = (r_1 + l_{n1}) \cos \theta_1 + (r_2 + l_{n2}) \cos \theta_2 \\ \quad \quad \quad \quad \quad \quad -Y_1 \sin \theta_1 + Y_2 \sin \theta_2 \\ d(s, Y_1, Y_2) = (r_1 + l_{n1}) \sin \theta_1 + (r_2 + l_{n2}) \sin \theta_2 \\ \quad \quad \quad \quad \quad \quad +Y_1 \cos \theta_1 - Y_2 \cos \theta_2 \end{cases} \quad (84)$$

Thus, let us define

$$\begin{cases} f_{id} = \frac{f_d}{r_1 + r_2} \{l \cos \theta_i + d \sin \theta_i\} \\ \lambda_{id} = (-1)^i \frac{f_d}{r_1 + r_2} \{-l \sin \theta_i + d \cos \theta_i\} \end{cases} \quad i = 1, 2 \quad (85)$$

and remark that they satisfy

$$f_{1d}\mathbf{n}_1 + f_{2d}\mathbf{n}_2 + \lambda_{1d}\mathbf{e}_1 + \lambda_{2d}\mathbf{e}_2 = 0 \quad (86)$$

Further, the right hand side can now be written as follows:

$$f_{1d}\mathbf{w}_1 + f_{2d}\mathbf{w}_2 + \lambda_{1d}\mathbf{w}_3 + \lambda_{2d}\mathbf{w}_4 = (0, 0, S_N)^T \quad (87)$$

where

$$\begin{aligned} S_N &= \frac{f_d}{r_1 + r_2} \left\{ - (l \cos \theta_1 + d \sin \theta_1) Y_1 \right. \\ & \quad \left. + (l \cos \theta_2 + d \sin \theta_2) Y_2 - (l \sin \theta_1 - d \cos \theta_1) l_{n1} \right. \\ & \quad \left. - (l \sin \theta_2 - d \cos \theta_2) l_{n2} \right\} \\ &= \frac{f_d}{r_1 + r_2} \left\{ l(r_1 \sin \theta_1 + r_2 \sin \theta_2) - d(r_1 \cos \theta_1 + r_2 \cos \theta_2) \right\} \end{aligned} \quad (88)$$

This shows according to (87) that, if  $S_N$  tends to vanish, then the force/torque balance is established, that is, the total sum of wrench vectors exerted to the object becomes zero.

## 6. Control Signal for Blind Grasping and a Morse-Lyapunov Function

From the practical standpoint of designing a control signal for stable grasping, it is important to see that objects to be grasped are changeable but the pair of robot fingers are always the same. That is, for designing control signals, we are unable to use physical parameters of the object such as the location  $(x_m, y_m)$  of its mass center and object geometry. On the contrary, it is possible to assume the knowledge of finger kinematics like finger link lengths and locations of the centers of finger-end spheres and to use measurement data of finger joint angles and angular velocities. In view of these points, let us propose a family of control signals defined by

$$u_i = -c_i \dot{q}_i + (-1)^i \frac{f_d}{r_1 + r_2} J_i^T(q_i) \begin{pmatrix} x_{01} - x_{02} \\ y_{01} - y_{02} \end{pmatrix} - r_i \hat{N}_i \bar{\mathbf{e}}_i, \quad i = 1, 2 \quad (89)$$

where

$$\hat{N}_i(t) = \gamma_i^{-1} r_i \bar{\mathbf{e}}_i^T (q_i(t) - q_i(0)) \quad (90)$$

and  $\bar{\mathbf{e}}_1 = (1, 1, 1)^T$ ,  $\bar{\mathbf{e}}_2 = (1, 1)^T$ ,  $c_i$  denotes a damping factor, and  $\gamma_i > 0$  a positive gain specified later.

The closed-loop dynamics of motion of the overall fingers-object system can be derived by substituting  $u_i$  of (89) into (75) for  $i = 1, 2$ . In order to spell out the dynamics in a more physically meaningful way for later discussions, first note the following three equalities:

$$\frac{1}{2} \left\{ (x_{01} - x_{02})^2 + (y_{01} - y_{02})^2 \right\} = \frac{1}{2} (l^2 + d^2) \quad (91)$$

$$f_{id} \frac{\partial Q_i}{\partial q_i} + \lambda_{id} \frac{\partial R_i}{\partial q_i} = (-1)^i \left[ J_i^T \begin{pmatrix} -f_d \cos(\theta + \theta_i) \\ -f_d \sin(\theta + \theta_i) \end{pmatrix} \right]$$

$$\begin{aligned}
& + \lambda_{id} \left( \begin{array}{c} \sin(\theta + \theta_i) \\ \cos(\theta + \theta_i) \end{array} \right) \left. \right\} - \lambda_{id} r_i \bar{\mathbf{e}}_i \Bigg\} \\
& = \frac{(-1)^i f_d}{r_1 + r_2} \left\{ -J_i^T R_\theta \left( \begin{array}{c} l \\ -d \end{array} \right) - \lambda_{id} r_i \bar{\mathbf{e}}_i \right\} \\
& = (-1)^i \frac{f_d}{r_1 + r_2} \left\{ J_i^T \left( \begin{array}{c} x_{01} - x_{02} \\ y_{01} - y_{02} \end{array} \right) - \lambda_{id} r_i \bar{\mathbf{e}}_i \right\}, \\
& \quad i = 1, 2 \quad (92)
\end{aligned}$$

Then, by substituting (87), (91), and (92) into equations (66) to (70), it is possible to express the closed-loop dynamics in the following forms:

$$I\ddot{\theta} - \Delta f_1 Y_1 + \Delta f_2 Y_2 - \Delta \lambda_1 l_{n1} + \Delta \lambda_2 l_{n2} - S_N = 0 \quad (93)$$

$$M \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} - \Delta f_1 \mathbf{n}_1(\theta) - \Delta f_2 \mathbf{n}_2(\theta) - \Delta \lambda_1 \mathbf{e}_1(\theta) - \Delta \lambda_2 \mathbf{e}_2(\theta) = 0 \quad (94)$$

$$\begin{aligned}
G_i \ddot{q}_i + \left\{ \frac{1}{2} \dot{G}_i + S_i \right\} \dot{q}_i + c_i \dot{q}_i + \Delta f_i J_i^T \mathbf{n}_i(\theta) \\
+ \Delta \lambda_i \left\{ J_i^T \mathbf{e}_i(\theta) + (-1)^i r_i \bar{\mathbf{e}}_i \right\} + r_i \left\{ (-1)^i \lambda_{id} + \hat{N}_i \right\} \bar{\mathbf{e}}_i = 0, \\
i = 1, 2 \quad (95)
\end{aligned}$$

Similarly to the form of (78), these equations can be written in the following way:

$$\begin{aligned}
G(\mathbf{X}) \ddot{\mathbf{X}} + \left\{ \frac{1}{2} \dot{G} + S + C \right\} \dot{\mathbf{X}} + \frac{\partial \Phi}{\partial \mathbf{X}} \Delta \lambda \\
+ S_N \mathbf{b}_\theta + \sum_{i=1,2} \left\{ \hat{N}_i + (-1)^i \lambda_{id} \right\} \mathbf{b}_i = 0 \quad (96)
\end{aligned}$$

where

$$\Delta \lambda = (f_1 - f_{1d}, f_2 - f_{2d}, \lambda_1 - \lambda_{1d}, \lambda_2 - \lambda_{2d})^T \quad (97)$$

$$\begin{cases} \mathbf{b}_\theta = (0, 0, 1, 0, 0, 0, 0, 0)^T \\ \mathbf{b}_1 = (0, 0, 0, 1, 1, 1, 0, 0)^T \\ \mathbf{b}_2 = (0, 0, 0, 0, 0, 0, 1, 1)^T \end{cases} \quad (98)$$

At this stage, it is important to note that in accordance with four constraints  $\Phi = 0$ , the velocity vector  $\dot{\mathbf{X}}$  belongs to the kernel space of  $(\partial \Phi / \partial \mathbf{X})^T$  and therefore  $\dot{\mathbf{X}}^T \partial \Phi / \partial \mathbf{X} = 0$ . Further by using (89) and taking inner product of (78) and  $\dot{\mathbf{X}}$ , we obtain

$$\begin{aligned}
\frac{d}{dt} \left\{ K + \frac{f_d}{2(r_1 + r_2)} \left\{ (x_{01} - x_{02})^2 + (y_{01} - y_{02})^2 \right\} + \sum_{i=1,2} \frac{\gamma_i}{2} \hat{N}_i^2 \right\} \\
= - \sum_{i=1,2} c_i \|\dot{q}_i\|^2 \quad (99)
\end{aligned}$$

where  $K$  denotes the system's kinetic energy defined by

$$K = \sum_{i=1,2} \frac{1}{2} \dot{q}_i^T G_i(q_i) \dot{q}_i + \frac{M}{2} (\dot{x}^2 + \dot{y}^2) + \frac{I}{2} \dot{\theta}^2 \quad (100)$$

The relation of (99) must be equivalently derived by taking inner product of  $\dot{\mathbf{X}}$  and the closed-loop dynamics of equation (96). To verify this, let us define

$$p_1 = q_{11} + q_{12} + q_{13} = \bar{\mathbf{e}}_1^T q_1, \quad p_2 = q_{21} + q_{22} = \bar{\mathbf{e}}_2^T q_2 \quad (101)$$

$$P = \frac{f_d}{2(r_1 + r_2)} \left\{ l^2(s, Y_1, Y_2) + d^2(s, Y_1, Y_2) \right\} + \sum_{i=1,2} \frac{\gamma_i}{2} \hat{N}_i^2 \quad (102)$$

where  $l$  and  $s$  are defined in (84). It should be remarked that, with the aid of expressions of integral form of rolling constraints shown in (63), we have

$$\begin{aligned}
\frac{\partial l}{\partial \theta} &= -\frac{\partial Y_1}{\partial \theta} \sin \theta_1 + \frac{\partial Y_2}{\partial \theta} \sin \theta_2 \\
&= -r_1 \sin \theta_1 - r_2 \sin \theta_2 \quad (103)
\end{aligned}$$

$$\frac{\partial d}{\partial \theta} = r_1 \cos \theta_1 + r_2 \cos \theta_2 \quad (104)$$

and hence

$$\begin{aligned}
\frac{\partial P}{\partial \theta} &= \frac{f_d}{r_1 + r_2} \left\{ l \frac{\partial l}{\partial \theta} + d \frac{\partial d}{\partial \theta} \right\} \\
&= \frac{f_d}{r_1 + r_2} \{-l(r_1 \sin \theta_1 + r_2 \sin \theta_2) + d(r_1 \cos \theta_1 + r_2 \cos \theta_2)\} \\
&= S_N \quad (105)
\end{aligned}$$

Similarly, from (63), (56), and (60) we have

$$\begin{aligned}
\frac{\partial P}{\partial p_i} &= \frac{f_d}{r_1 + r_2} \left\{ l \frac{\partial l}{\partial p_i} + d \frac{\partial d}{\partial p_i} \right\} + \gamma_i \hat{N}_i \frac{\partial \hat{N}_i}{\partial p_i} \\
&= (-1)^i \frac{f_d}{r_1 + r_2} \left\{ l \frac{\partial Y_i}{\partial p_i} \sin \theta_i - d \frac{\partial Y_i}{\partial p_i} \cos \theta_i \right\} + \gamma_i \hat{N}_i \frac{\partial \hat{N}_i}{\partial p_i} \\
&= \frac{-r_i f_d}{r_1 + r_2} \{ l \sin \theta_i - d \cos \theta_i \} + r_i \hat{N}_i \\
&= (-1)^i r_i \lambda_{id} + r_i \hat{N}_i, \quad i = 1, 2 \quad (106)
\end{aligned}$$

Thus, the inner product of  $\dot{\mathbf{X}}$  and (96) or the relation of (99) is reduced to

$$\frac{d}{dt} \{K + P\} = - \sum_{i=1,2} c_i \|\dot{q}_i\|^2 \quad (107)$$

In other words, the closed-loop dynamics of (96) can be expressed in the form

$$G(\mathbf{X}) \ddot{\mathbf{X}} + \left\{ \frac{1}{2} \dot{G} + S + C \right\} \dot{\mathbf{X}} + \frac{\partial P}{\partial \mathbf{X}} + \frac{\partial \Phi}{\partial \mathbf{X}} \Delta \lambda = 0 \quad (108)$$

This is interpreted as a Lagrange equation of the Lagrangian

$$L = K - P - \Phi \Delta \lambda \quad (109)$$

in accompany with the external damping torques  $c_i \dot{q}_i$  for  $i = 1, 2$  through finger joints. The scalar function  $P$  defined by equation (102) is a quadratic function of  $Y_1, Y_2, p_1$ , and  $p_2$  and hence it is regarded as a quadratic function of  $\theta, p_1$ , and  $p_2$  since  $Y_i$  can be regarded as a linear function of  $\theta$  and  $p_i$  for  $i = 1, 2$  because of (63), (56), and (60). Hence,  $P$  can be regarded as a Morse function defined on the Riemannian submanifold induced by four constraints described by  $Q_i = 0$  and  $R_i = 0$  for  $i = 1, 2$ .

## 7. Physical Insights into Gradient and Hessian of the Morse Function

The physical meaning of control signals for blind grasping defined by (89) is quite simple. The first term of the right hand side of (89) plays a role of damping for rotational motion of finger joints. Damping for motion of the object is exerted from velocity constraints  $\dot{Q}_i = 0$  and  $\dot{R}_i = 0$  for  $i = 1, 2$  as discussed in detail in the previous paper [1]. The second term plays a role of fingers-thumb opposition that induces minimization of the distance between  $O_{01}$  and  $O_{02}$ , centers of finger-end spheres. The distance is equivalent to  $\sqrt{l^2 + d^2}$  as discussed in section 5. The third term plays an important role in suppressing excess movements of rotation of finger joints. These characteristics of the control signal condense into the Lagrange equation of motion with 1) the potential  $P(s, \theta, p_1, p_2)$  of (102), 2) the lifting

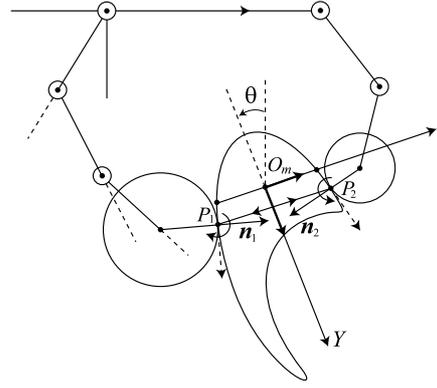
Table 1 Hessian matrix of the potential  $P$ .

$$\begin{aligned} \frac{\partial^2 P}{\partial \theta^2} &= \frac{\partial S_N}{\partial \theta} = \frac{f_d}{r_1+r_2} \left\{ \frac{\partial l}{\partial \theta} (r_1 \sin \theta_1 + r_2 \sin \theta_2) \right. \\ &\quad \left. - \frac{\partial d}{\partial \theta} (r_1 \cos \theta_1 + r_2 \cos \theta_2) \right\} \\ &= \frac{f_d}{r_1+r_2} \left\{ (r_1 \sin \theta_1 + r_2 \sin \theta_2)^2 + (r_1 \cos \theta_1 + r_2 \cos \theta_2)^2 \right\} \\ &= \frac{f_d}{r_1+r_2} \left\{ r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) \right\} \quad (T-1) \\ \frac{\partial^2 P}{\partial p_i^2} &= r_i^2 \left\{ \frac{f_d}{r_1+r_2} + \frac{1}{\gamma_i} \right\}, \quad i = 1, 2 \quad (T-2) \\ \frac{\partial^2 P}{\partial p_i \partial \theta} &= -\frac{f_d}{r_1+r_2} \left\{ r_i^2 + r_1 r_2 \cos(\theta_1 - \theta_2) \right\}, \quad i = 1, 2 \quad (T-3) \\ \frac{\partial^2 P}{\partial p_2 \partial p_1} &= -\frac{r_1 r_2 f_d}{r_1+r_2} \cos(\theta_1 - \theta_2) \quad (T-4) \end{aligned}$$

$(\partial \Phi / \partial X) \lambda_d$ , where  $\lambda_d = (f_{1d}, f_{2d}, \lambda_{1d}, \lambda_{2d})^T$ , and 3) the gradient  $\partial P / \partial X$  of the potential. In other words, equation (108) implies that if the artificial potential  $P$  attains its minimum and, at the same time, makes the gradient  $\partial P / \partial X$  vanish at some  $s = s^*$  ( $s = (s_1, s_2)^T$ ), and, moreover, the artificial potential were positive definite in  $X$  under the four constraints, then the equilibrium position that minimizes  $P$  would be asymptotically stable. Unfortunately,  $P$  is a quadratic function with respect to only  $\theta$ ,  $p_1$ , and  $p_2$  and therefore  $P$  is only nonnegative definite in  $X$ . Nevertheless, it is possible to show that the Hessian matrix of  $P$  with respect to  $\theta$ ,  $p_1$ , and  $p_2$  becomes positive definite in these three variables as shown in Table 1 that is calculated by partially differentiating the gradients  $\partial P / \partial \theta$ ,  $\partial P / \partial p_1$ , and  $\partial P / \partial p_2$  of equations (105) to (106) with respect to  $\theta$ ,  $p_1$ , and  $p_2$  again. Apparently, the Hessian becomes positive definite provided that  $\gamma_i$  is chosen as being of similar order of  $(r_i / f_d)$ , and  $\theta_i(s_i)$  remains in a region such that  $|\theta_i(s_i)| < \pi/6$  for  $i = 1, 2$ . Then, by using a similar argument used in proving the stability on a submanifold for DOF-redundant systems [8],[9], a solution trajectory to the Lagrange equation converges exponentially to the constraint equilibrium manifold. In this paper we omit the details of the argument, but presents a physical meaning of the gradient  $\partial P / \theta$  together with  $\partial P / \partial p_i$  ( $i = 1, 2$ ).

As shown in Fig. 7 all rotational motions of the object emerge around axes that are perpendicular to the  $xy$ -plane. Here, we consider the rotational moment that may emerge at the contact point  $P_2$  exerted by the pressing force  $f_1$  to the object from the other contact point  $P_1$ . Another force  $f_2$  at  $P_2$  that presses the object generates the torque around the  $z$ -axis at  $P_1$ . The sum of these torques around the  $z$ -axis can be expressed as

$$\begin{aligned} \mathbf{r}_1 \times \mathbf{f}_1 + \mathbf{r}_2 \times \mathbf{f}_2 &= \overrightarrow{P_1 P_2} \times \frac{r_1 f_d}{r_1+r_2} \mathbf{n}_1 + \overrightarrow{P_2 P_1} \times \frac{r_2 f_d}{r_1+r_2} \mathbf{n}_2 \\ &= \begin{pmatrix} X(s_1) - X(s_2) \\ Y(s_1) - Y(s_2) \\ 0 \end{pmatrix} \times \frac{r_1 f_d}{r_1+r_2} \begin{pmatrix} \cos \theta_1 \\ -\sin \theta_1 \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} X(s_2) - X(s_1) \\ Y(s_2) - Y(s_1) \\ 0 \end{pmatrix} \times \frac{r_2 f_d}{r_1+r_2} \begin{pmatrix} -\cos \theta_2 \\ \sin \theta_2 \\ 0 \end{pmatrix} \end{aligned}$$

Fig. 7 The pressing force to the object at  $P_1$  in the direction  $\mathbf{n}_1$  induces a rotational moment around  $P_2$  and vice versa.

$$= \frac{f_d}{r_1+r_2} \begin{pmatrix} 0 \\ 0 \\ -(X_1 - X_2)(r_1 \sin \theta_1 + r_2 \sin \theta_2) \\ -(Y_1 - Y_2)(r_1 \cos \theta_1 + r_2 \cos \theta_2) \end{pmatrix} \quad (110)$$

where we denote  $X_i = X(s_i)$  and  $Y_i = Y(s_i)$  for abbreviation. We see also that, from geometric relations of the vectors  $\overrightarrow{O_m P_i}$ , and quantities  $Y_i$ , and  $l_{ni}$ , expressed in local coordinates  $O_m-XY$  (see Fig. 5) it follows that

$$\begin{aligned} \begin{pmatrix} l \\ -d \end{pmatrix} &= r_1 \begin{pmatrix} \cos \theta_1 \\ -\sin \theta_1 \end{pmatrix} + r_2 \begin{pmatrix} \cos \theta_2 \\ -\sin \theta_2 \end{pmatrix} \\ &\quad + R_{\theta_1} \begin{pmatrix} l_{n1} \\ -Y_1 \end{pmatrix} + R_{\theta_2} \begin{pmatrix} l_{n2} \\ -Y_2 \end{pmatrix} \\ &= r_1 \begin{pmatrix} \cos \theta_1 \\ -\sin \theta_1 \end{pmatrix} + r_2 \begin{pmatrix} \cos \theta_2 \\ -\sin \theta_2 \end{pmatrix} \\ &\quad - \begin{pmatrix} X(s_1) - X(s_2) \\ Y(s_1) - Y(s_2) \end{pmatrix} \quad (111) \end{aligned}$$

Applying this relation to (110), we see that

$$\begin{aligned} \mathbf{r}_1 \times \mathbf{f}_1 + \mathbf{r}_2 \times \mathbf{f}_2 &= \frac{f_d}{r_1+r_2} \begin{pmatrix} 0, 0, l \sum_{i=1,2} r_i \sin \theta_i - d \sum_{i=1,2} r_i \cos \theta_i \end{pmatrix}^T \\ &= (0, 0, S_N)^T \quad (112) \end{aligned}$$

As a summary of the argument, it is possible to conclude that a solution trajectory to (108) converges asymptotically to some equilibrium state  $\mathbf{X} = \mathbf{X}^*$  with some  $s = s^*$  with  $\partial P / \partial X = 0$  at  $\mathbf{X} = \mathbf{X}^*$  and  $s = s^*$ .

## 8. Conclusions

A natural extension of a hybrid position/force control for robots with redundant degrees-of-freedom is presented by an extended Riemannian geometric approach. It is shown that any supply of the constant pressing force to the environment through joint actuations is not relevant to motions in the kernel space of the gradient of the constraint. An extension of problems of grasping and manipulation of rigid objects to the case of 2-dimensional objects with arbitrary shape is also treated from the viewpoint of Dirichlet-Lagrange's stability by introducing a non-negative definite Morse function on a Riemannian submanifold together with a damping shaping.

It should be remarked that, if an object to be grasped and manipulated by a pair of robot fingers is of arbitrary shape, constraint submanifolds and tangent bundles must be parametrized

by arclength parameters of the object contours. Therefore, the Euler equation must be accompanied with a set of first-order differential equations for updating the arclength parameters, in which curvatures of the contours appear as quantities of the second-order fundamental form.

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### Appendix

Suppose that the contact point  $P_1$  on the left hand fingerend with the object contour moves along the fingerend circle starting from the initial time  $t = 0$  to the present  $t$  (Fig. 6). The arclength of its movement can be expressed by

$$\begin{aligned} -\{s_1(t) - s_1(0)\} &= r_1\{\varphi_1(t) - \varphi_1(0)\} \\ &= r_1[\{\theta(t) + \theta_1(s_1(t)) - p_1(t)\} - \{\theta(0) + \theta_1(s_1(0)) - p_1(0)\}] \end{aligned} \quad (\text{A. 1})$$

Taking the derivative of both sides of (A. 1) and denoting the curvature of the object contour by  $\kappa_1(s)$  that is equal to  $d\theta_1(s)/ds$ , we have

$$-\frac{ds_1}{dt} = r_1(\dot{\theta} + \dot{p}_1) + \kappa_1(s_1)\frac{ds_1}{dt} \quad (\text{A. 2})$$

which is reduced to (76).

In light of (A. 2) and (55), we define a scalar function

$$R_1(t, s_1) = r_1\{\theta + \theta_1(s_1) - p_1\} + s_1 + Y_1 - l_{e1}(s_1) \quad (\text{A. 3})$$

It is then easy to see that

$$\begin{aligned} \frac{d}{dt}l_{e1} &= \frac{ds_1}{dt} - l_{n1}(s_1)\frac{d\theta_1}{ds_1}\frac{ds_1}{dt} \\ \frac{d}{dt}Y_1 &= (\dot{x}_{01} - \dot{x})\sin(\theta + \theta_1) + (\dot{y}_{01} - \dot{y})\cos(\theta + \theta_1) \end{aligned} \quad (\text{A. 4})$$

$$\begin{aligned} &-(r_1 + l_{n1})\left(\dot{\theta} + \frac{d\theta_1}{ds_1}\frac{ds_1}{dt}\right) \\ &= r_1\dot{p}_1 + l_{n1}\dot{\theta} - (r_1 + l_{n1})\left(\dot{\theta} + \frac{d\theta_1}{ds_1}\frac{ds_1}{dt}\right) \end{aligned} \quad (\text{A. 5})$$

Thus, substituting (A. 4) and (A. 5) into the time derivative of  $R_1$  of (A. 3), we see that

$$\frac{d}{dt}R_1 = \frac{\partial R}{\partial t} + \frac{\partial R}{\partial s_1}\frac{ds_1}{dt} = 0 \quad (\text{A. 6})$$

Hence, it is possible to define  $R_1(t) = \tilde{R}_1(t) - \tilde{R}_1(0)$  and thereby regard the equality  $R_1(t) = 0$  as a holonomic constraint. Note that differentiation of  $R_1(t)$  in  $t$  under fixed  $s_1$  yields (59).

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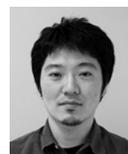
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