Modeling and Control of a Pair of Robot Fingers with Saddle Joint under Orderless Actuations

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Abstract—A new robot hand dynamics model with rolling constraints and with a saddle joint at one finger is proposed, where two saddle-joint actuations are considered to be orderless. Spinning motion around the opposition axis connecting two center points of each finger-tip contact area with an object is faithfully treated, and a viscosity model for damping rotational motion of the object is proposed. A class of control signals without referring to object kinematics or using external sensing is proposed. Finally, numerical simulation results show the stability of motion of the overall closed-loop dynamics supplied with the proposed control input.

I. INTRODUCTION

The whole stimuli to the human brain, which arise from making and using tools by hands, have accelerated the evolutionary speed of the human brain. The primatologist Napier [1] claims that most of important movements of the hands are based upon finger-thumb opposition shown in Fig.1, that has contributed to the progress of humanity. Despite such a profound proposition, there is a dearth of robotic researches paying much attention to consideration of control functions of multi-fingered hands.

For the sake of raveling mysteries of the human hand and realizing its dexterous movements, many robotic researchers devote a lot of their elaborate efforts to the research of robot hands [2] [3] [4]. However, most of the researchers were attracted by kinematics and planning of motions realizing force/torque closure for stable grasp by robot fingers in a static sense. On the other hand, rolling geometry between two arbitrarily-shaped objects is rigorously discussed [5] [6]. However, all the researches have remained in a kinematic or semi-dynamic meaning. In the field of multibody dynamics [7], many models with constraints in 3-D space are presented without modeling physically faithful interaction between a robot finger and an object surface through 3-D rolling. An explicit 3-D dynamical pinching model with rolling constraints between a finger end and an object surface has been missing.

Around the year of 2000, Arimoto et al. first showed that 2-D object pinching is realized by using a pair of robot fingers with hemispherical ends, succeeding at embedding rolling constraints into the overall fingers-object dynamics [8] [9]. The degrees-of-freedom redundancy problem of the overall fingers-object dynamics for desired tasks is overcome by proposing a stability concept called “Stability on a
manifold”. In the year of 2006 [10], modeling of motion of pinching an object was extended from two-dimensional to three-dimensional pinching. In the modeling of pinching it is assumed that spinning motion of the object around the opposition axis connecting two contact points between the left or right finger end and the object surface has ceased and will not arise any more due to dry friction and micro-deformations near the two contact points. This assumption leads to a five-variables model of the object dynamics. In the year of 2008 [11] [12] [13], instead of this assumption, the modeling problem was posed as a more faithful model considering that spinning motion of the object is possible to arise but viscosity friction is exerted on \( x \) axis in the frame coordinates \( O-xyz \) (see Figure 2), which means the overall fingers-object dynamics as a full variables model is derived. Furthermore, in the previous paper [12], two saddle-joint motors of the robot finger are actuated in sequence. However, the saddle joint of the human thumb must be actuated around the two axes respectively without making any order [14] (see Figure 3).

In this paper, we propose a saddle joint model with the orderless relation like the human thumb, and a robot hand model with the saddle joint through taking account of rolling constraints that are Pfaffian. A pair of robot fingers with hemispherical tips made from soft material is planar with three actuators rotational around \( z \)-axis. The modeling of the right robot finger with a saddle joint which has orderless actuations

![Diagram](image)

**Fig. 4.** Modeling of the right robot finger with a saddle joint which has orderless actuations

on the frame coordinates \( O-xyz \) (see Figure 2). It is well known that the \( 3 \times 3 \) rotation matrix

\[
R_{21}(t) = (r_{X21}, r_{Y21}, r_{Z21}) \in SO(3) \tag{1}
\]

which is subject to the first-order differential equation representing infinitesimal rotation as follows:

\[
\frac{d}{dt} R_{21}(t) = R_{21}(t) \Omega_{21}(t) \tag{2}
\]

where

\[
\Omega_{21}(t) = \begin{pmatrix}
0 & -\dot{q}_{21z} & 0 \\
\dot{q}_{21z} & 0 & -\dot{q}_{21x} \\
0 & \dot{q}_{21x} & 0
\end{pmatrix} \tag{3}
\]

The equations expressing the mass center \( c.m. \) of the second link are derived by considering the relationship between the first link’s posture and the angle \( q_{22} \) which rotates around \( Z_{21} \) as follows:

\[
r_{X22} = \cos q_{22} r_{X21} + \sin q_{22} r_{Y21} \tag{4}
\]

\[
r_{Y22} = -\sin q_{22} r_{X21} + \cos q_{22} r_{Y21} \tag{5}
\]

\[
r_{Z22} = r_{Z21} \tag{6}
\]

The angular velocity \( \dot{q}_{22} = (\dot{q}_{22x}, \dot{q}_{22y}, \dot{q}_{22z})^T \) of the second link, which rotates around \( Z_{21} \), can be represented by \( \dot{q}_{22} = \dot{q}_{22x} r_{X21} + \dot{q}_{22y} r_{Y21} \). Therefore the mass center \( c.m. \) defined by \( x_{m22} = (x_{m22}, y_{m22}, z_{m22}) \) can be represented by \( x_{m22} = l_2 r_{Y21} + s_2 r_{Y22} \).

### III. Dynamics

The model of pinching a rigid object with parallel flat surfaces by two robot fingers with 3 DOFs and 3 DOFs is schematically shown in Fig.2. The left finger (finger \( i = 1 \)) is planar with three actuators rotational around \( z \)-axis. The
vector \( q_{11} = (q_{11}, q_{12}, q_{13})^T \) denotes the joint angles of the left hand side finger. In the previous paper [10], it is also assumed that spinning against the opposition axis is possible to arise but viscosity damps rotational motion of the object around \( z \)-axis, that is, about \( \omega_z \), where \( \omega = (\omega_x, \omega_y, \omega_z)^T \) denotes the vector of rigid body rotation in terms of frame coordinates \( O-xyz \) (see Figure 2). This viscosity model is effective but rough in a physical sense. Instead of it, we introduce a more faithful viscosity model which is exerted on rotational motion of the object around the \( r_X \) vector at two contact points between the left or right finger tip and the object surface as shown in Fig.5.

At the same time, we introduce the cartesian coordinates \( O_{c.m.}-XYZ \) fixed at the object frame and denote the three orthogonal unit vectors at the object frame in each corresponding direction \( X, Y, \) and \( Z \) by \( r_X = (r_{Xx}, r_{Xy}, r_{Xz})^T \), \( r_Y = (r_{Yx}, r_{Yy}, r_{Yz})^T \), and \( r_Z = (r_{Zx}, r_{Zy}, r_{Zz})^T \) as shown in Fig.2. Next, let us denote the cartesian coordinates of the object mass center \( O_{c.m.} \) by \( x = (x, y, z)^T \) based on the frame coordinates \( O-xyz \) and note that three mutually orthogonal unit vectors fixed at the object frame may rotate dependently on the angular velocity vector \( \omega \) of body rotation. Then, it is well known that the \( 3 \times 3 \) rotation matrix

\[
R(t) = (r_X, r_Y, r_Z) \in SO(3)
\]

is subject to the first-order differential equation

\[
\frac{d}{dt} R(t) = R(t) \Omega(t)
\]

where

\[
\Omega(t) = \begin{pmatrix}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & -\omega_x \\
-\omega_y & \omega_x & 0
\end{pmatrix}
\]

Next, denote the position of the center of each hemispherical finger-end by \( x_{0i} = (x_{0i}, y_{0i}, z_{0i})^T \). Then, it is possible to notice that (see Fig.5)

\[
x_i = x_{0i} - (-1)^i(r_i - \Delta x_i)r_X
\]

\[
x = x_{0i} - (-1)^i(r_i - \Delta x_i + l_i)r_X - Y_i r_Y - Z_i r_Z
\]

Since each contact point \( O_i \) can be expressed by the coordinates \((-1)^i l_i, Y_i, Z_i\) based on the object frame \( O_{c.m.-XYZ} \), taking an inner product between Equation (11) and \( r_Y \) gives rise to

\[
Y_i = (x_{0i} - x)^T r_Y, \ i = 1, 2
\]

Similarly, it follows that

\[
Z_i = (x_{0i} - x)^T r_Z, \ i = 1, 2
\]

A rolling constraint between one finger-end and its contacted object surface can be expressed by the equality of two contact point velocities expressed on either of finger-end spheres and its corresponding tangent plane (that is coincident with one of object flat surfaces) as follows [10]:

\[
\begin{cases}
\{r_1 - \Delta x_1\} \{\omega_z - r_{Z2}(q_{11} + \dot{q}_{12} + \dot{q}_{13})\} = \dot{Y}_1 \\
\{r_1 - \Delta x_1\} \{-\omega_y + r_Y(q_{11} + \dot{q}_{12} + \dot{q}_{13})\} = \dot{Z}_1 \\
\{r_2 - \Delta x_2\} \{-\omega_z + r_{Z2}(q_{11} + \dot{q}_{12} + \dot{q}_{13}) + r_{22}^T r_{Z22}\} = \dot{Y}_2 \\
\{r_2 - \Delta x_2\} \{-\omega_y + r_Y(q_{11} + \dot{q}_{12} + \dot{q}_{13}) - r_{Y2} r_{Z2} - r_{Z22}^T r_{Z22}\} = \dot{Z}_2
\end{cases}
\]

(14)

The rolling constraint conditions expressed through Equations (14) and (15) are non-holonomic but linear and homogeneous with respect to velocity variables. Hence, Equations (14) and (15) can be treated as Pfaffian constraints [2] that can be expressed with accompanying Lagrange’s multipliers \( \{\lambda_{Y1}, \lambda_{Z1}\} \) for Equation (14), and \( \{\lambda_{Y2}, \lambda_{Z2}\} \) for Equations (15) in such forms as

\[
\begin{cases}
\lambda_{Y1} \{Y_{T}^T q_{1} + Y_{T}^T x + Y_{\psi1} \dot{\psi} + Y_{\phi1} \dot{\phi} + Y_{\theta1} \dot{\theta}\} = 0 \\
\lambda_{Z1} \{Z_{T}^T q_{1} + Z_{T}^T x + Z_{\psi1} \dot{\psi} + Z_{\phi1} \dot{\phi} + Z_{\theta1} \dot{\theta}\} = 0
\end{cases}
\]

\[
i = 1, 2
\]

where

\[
\begin{align*}
Y_{q1} &= \frac{\partial Y_{T}}{\partial q_{1}} - (r_1 - \Delta x_1) \{(-1)^i r_{Y2} e_{1z} + r_{Z2} e_{ix}\} \\
&\quad + r_{T21}^T r_{Z21} e_{z1} \\
Y_{x1} &= \frac{\partial Y_{T}}{\partial x_{1}} \quad Y_{\psi1} = \frac{\partial Y_{T}}{\partial \psi_{1}} \\
&\quad + (-1)^i (r_1 - \Delta x_1) \\
Y_{\theta1} &= \frac{\partial Y_{T}}{\partial \theta_{1}}
\end{align*}
\]

and

\[
\begin{align*}
Z_{q1} &= \frac{\partial Z_{T}}{\partial q_{1}} - (r_1 - \Delta x_1) \{(-1)^i r_{Y2} e_{1z} + r_{Z2} e_{ix}\} \\
&\quad + r_{T21}^T r_{Z21} e_{z1} \\
Z_{x1} &= \frac{\partial Z_{T}}{\partial x_{1}} \quad Z_{\psi1} = \frac{\partial Z_{T}}{\partial \psi_{1}} \\
&\quad + (-1)^i (r_1 - \Delta x_1) \\
Z_{\theta1} &= \frac{\partial Z_{T}}{\partial \theta_{1}}
\end{align*}
\]

and \( q_1 = (q_{11}, q_{12}, q_{13})^T \), \( q_2 = (q_{21}, q_{22}, q_{23})^T \), \( e_{1z} = (1, 1, 0)^T \), \( e_{z1} = (0, 1, 0)^T \), \( e_{x1} = (0, 0, 1)^T \), and \( e_{12} = 0_3 \), and \( e_{22} = (0, 0, 1) \). To simplify notations, we rewrite Equation (16) into

\[
\begin{cases}
\lambda_{Y1} (Y_{T}) (dx/dt) = 0 \\
\lambda_{Z1} (Z_{T}) (dx/dt) = 0
\end{cases}
\]

\[
i = 1, 2
\]

where

\[
X = (q_{1}, q_{2}, x, \varphi, \phi, \theta)^T
\]

\[
\begin{align*}
Y_1 &= \{Y_{T}^T q_{1}, 0_2, Y_{T}^T x_1, Y_{\varphi1}, Y_{\phi1}, Y_{\theta1}\}^T \\
Y_2 &= \{0, Y_{T}^T q_{2}, Y_{T}^T x_2, Y_{\varphi2}, Y_{\phi2}, Y_{\theta2}\}^T
\end{align*}
\]

(18)
\[
\begin{align*}
Z_1 &= \left( Z_{Tq1}, 0, Z_{Tx1}, Z_{\varphi 1}, Z_{\phi 1}, Z_{\theta 1} \right)_T \\
Z_2 &= \left( 0, Z_{Tq2}, Z_{Tx2}, Z_{\varphi 2}, Z_{\phi 2}, Z_{\theta 2} \right)_T
\end{align*}
\] (19)

The reproducing force due to finger-tip deformation can be described as (see [13])

\[
f_i(\Delta x_i, \Delta \dot{x}_i) = \bar{f}_i(\Delta x_i) + \xi_i(\Delta x_i) \Delta \dot{x}_i
\] (20)

where

\[
\bar{f}_i(\Delta x_i) = k_i \Delta x_i^2, \quad i = 1, 2
\] (21)

with stiffness parameter \(k_i > 0[N/m^2]\) and \(\xi_i(\Delta x_i)\) is a positive scalar function of \(\Delta x_i\).

The viscosities around \(r_X\) at two contact points due to spinning motion around the opposing axis is represented by the following Rayleigh's dissipation functions:

\[
R_{t1} = \frac{c_{t1}}{2} \| \{ \omega - (\dot{q}_{11} + \dot{q}_{12} + \dot{q}_{13}) e_x \} r_X \|^2
\] (22)

\[
R_{t2} = \frac{c_{t2}}{2} \| \{ \omega - \dot{q}_{21} e_z - \dot{q}_{21} e_x - \dot{q}_{22} \} r_X \|^2
\] (23)

\[
R_t = R_{t1} + R_{t2}
\] (24)

where \(R_{t1}\) expresses the Rayleigh’s dissipation function at the left contact point, \(R_{t2}\) at the right contact point, both \(c_{t1}\) and \(c_{t2}\) positive constant values, \(e_x = (1, 0, 0)^T\) and \(e_z = (0, 0, 1)^T\). The function \(R_t\) is derived from taking into considerations that the difference between the rotational velocity of motion of the robot fingers and the rotational velocity of motion of the object relative to the \(r_X\) vector generates viscosity, and no viscosity arises in the case of equal velocities between the velocity of each robot finger and the velocity of the object relative to the \(r_X\) vector.

The Lagrangian for the overall fingers-object system can be expressed by the scalar quantity \(L = K - P\), where \(K\) denotes the total kinetic energy expressed as

\[
K = \frac{1}{2} q_i^T H_1 q_i + \frac{1}{2} m_{21} \dot{x}_{m21}^T \dot{x}_{m21} + \frac{1}{2} \dot{q}_{21}^T H_2 \dot{q}_{21}
+ \frac{1}{2} \dot{x}_{m22}^T \dot{x}_{m22}
+ \frac{1}{2} \left( \dot{q}_{21} + \dot{q}_{22} \right)^T \tilde{H}_{22} \left( \dot{q}_{21} + \dot{q}_{22} \right)
+ \frac{1}{2} M \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + \frac{1}{2} \omega^2 T H_0 \omega
\] (25)

and \(P\) denotes the total potential energy expressed as

\[
P = P_1 + P_2 - Mg + \sum_{i=1,2} P_{\Delta x_i}
\] (26)

where \(H_1\) stands for the inertia matrix for finger 1, \(m_{21}\) the mass of the first link of the finger 2, \(m_{22}\) the mass of the second link of it, \(M\) the mass of the object, \(P_i\) the potential energy of finger \(i\), \(g\) the gravity constant, and \(P_{\Delta x_i} = \int_0^{\Delta x_i} \bar{f}_i(\xi) d\xi\) the potential energy of reproducing force for finger \(i\), and \(H_0\) is given in the following:

\[
H_0 = R(t)HR^T(t)
\] (27)

where \(H\) stands for the constant inertia matrix of the object that must be evaluated on the basis of fixed body coordinates \(O_{c.m.}-XYZ\), that is,

\[
H = \begin{pmatrix}
I_{XX} & I_{XY} & I_{XZ} \\
I_{YX} & I_{YY} & I_{YZ} \\
I_{ZX} & I_{ZY} & I_{ZZ}
\end{pmatrix}
\] (28)

Similarly, \(H_{2i}(i = 1, 2)\) is given in the following:

\[
H_{2i} = R_{2i}(t)\tilde{H}_{2i} R_{2i}^T(t)
\] (29)

where \(\tilde{H}_{2i}\) similarly stands for the constant inertia matrix of the first link of the robot finger 2 and \(H_{22}\) the constant inertia matrix of its second link, that must be expressed on the basis of fixed robot link coordinates \(O_{c.m.21}-X_{21}Y_{21}Z_{21}\) respectively, that is,

\[
\tilde{H}_{2i} = \begin{pmatrix}
I_{XX2i} & I_{XY2i} & I_{XZ2i} \\
I_{YX2i} & I_{YY2i} & I_{YZ2i} \\
I_{ZX2i} & I_{ZY2i} & I_{ZZ2i}
\end{pmatrix}, \quad i = 1, 2
\] (30)

where

\[
R_{22}(t) = (r_{X22}, r_{Y22}, r_{Z22}) \in SO(3)
\] (31)

Thus, owing to the variational principle applied to the form

\[
- \int_{t_0}^{t_1} \delta L dt = \int_{t_0}^{t_1} \sum_{i=1,2} \left( u_i^T \delta q_i - \left( \lambda_{Y_i} Y_i^T + \lambda_{Z_i} Z_i^T \right) \delta X \right) dt
\]

\[
- \int_{t_0}^{t_1} \left( \frac{\partial R_t}{\partial X} \delta X + \sum_{i=1,2} \xi_i(\Delta x_i) \Delta \dot{x}_i \frac{\partial \Delta \dot{x}_i}{\partial X} \delta X \right) dt
\] (32)

we obtain a set of Lagrange’s equations of motion of the overall system:

\[
\begin{align*}
H_i q_i + \left( \frac{1}{2} \delta H_t + S_i \right) \dot{q}_i + \frac{\partial R_t}{\partial q_i} \dot{q}_i - (-1)^i f_i \left( t_0 \right) r_X \\
- \lambda_{Y_i} Y_i q_i - \lambda_{Z_i} Z_i q_i + g_i = u_i, \quad i = 1, 2
\end{align*}
\] (33)

\[
M \ddot{x} - (f_1 - f_2) r_X + (\lambda_{Y1} + \lambda_{Y2}) r_Y
\]

\[
+ (\lambda_{Z1} + \lambda_{Z2}) r_Z - Mg = 0
\] (34)

\[
H_{00} \ddot{\omega} + \left( \frac{1}{2} \delta H_0 + S \right) \omega + \frac{\partial R_t}{\partial \omega} \omega + f_1 \left( -Z_1 \right)
\]

\[
+ f_2 \left( Z_2 - Y_1 \right) - \lambda_{Y1} \left( Z_1 \right) - \lambda_{Y2} \left( 0 \right) - l_2 - l_2
\]

\[
- \lambda_{Z1} \left( -Y_1 \right) - \lambda_{Z2} \left( -Y_2 \right) = 0
\] (35)

where calculation of partial differentiations of \(Y_i, Z_i\) in \(\varphi, \psi, \theta\) is presented in Table I. Further, it is possible to see

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TABLE I
PARTIAL DERIVATIVES OF CONSTRAINTS IN (ϕ, ψ, θ).

\[
\begin{align*}
\frac{\partial p_{\Delta x_i}}{\partial \phi} &= \frac{\partial p_{\Delta x_i}}{\partial \psi} = 0, \\
\frac{\partial y_i}{\partial \phi} &= \left( x_0i - x \right)^T r_x = 0, \\
\frac{\partial s_i}{\partial \phi} &= \left( x_0i - x \right)^T r_y = -r_i. \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial z_{\Delta x_i}}{\partial \phi} &= \left( x_0i - x \right)^T r_z = -\left( -1 \right)^i (r_1 - \Delta x_i + l_i) \\
\frac{\partial z_{\Delta x_i}}{\partial \psi} &= \left( x_0i - x \right)^T \hat{r}_z = -\left( -1 \right)^i (r_1 - \Delta x_i + l_i) \\
\frac{\partial z_{\Delta x_i}}{\partial \theta} &= \left( x_0i - x \right)^T \hat{r}_z = 0, \\
Y_{\Delta x_i} &= \frac{\partial \theta}{\partial \phi}, \\
Z_{\Delta x_i} &= \frac{\partial \theta}{\partial \psi}, \\
\end{align*}
\]

\[
\begin{align*}
\sum q_i^T u_i &= \frac{d}{dt} \left( K + P \right) + 2R_i + \sum_{i=1,2} \xi(\Delta x_i) \Delta x_i^2 \\
\end{align*}
\]

From taking inner products between \( \dot{q}_i \) and Equation (33), \( \dot{x} \) and Equation (34), and \( \omega \) and Equation (35), we obtain the relation

\[
\sum_{i=1,2} q_i^T u_i = \frac{d}{dt} \left( K + P \right) + 2R_i + \sum_{i=1,2} \xi(\Delta x_i) \Delta x_i^2
\]

Finally, we interestingly notice six wrench vectors composed of accompaniments to \( f_1, f_2, \lambda_1, \lambda_2, \lambda_{21}, \) and \( \lambda_{22} \) in Equations (34) and (35). The six wrench vectors and the external force composed of the last term in Equation (34) work on the 3-dimensional object.

IV. CONTROL SIGNALS

A control input realizing stable grasping under the gravity effect without using object kinematics or external sensing, which is called “blind grasping”, is proposed as follows:

\[
\begin{align*}
\dot{q}_i &= -C_i \dot{q}_i + \left( -1 \right)^i \frac{f_1}{r_1 + r_2} J^T_{0i} (x_{0i} - x_{02}) \\
&\quad - \frac{M g}{2} \left( \frac{\partial y_i}{\partial q_i} - r_i \dot{N}_{\Delta x_i} e_{\Delta x_i} \\
&\quad - r_i \dot{N}_{\Delta x_i} e_{\Delta x_i}, \quad i = 1, 2 \right)
\end{align*}
\]

where

\[
\begin{align*}
\dot{M} &= \dot{M}(0) + \int_0^t \frac{\gamma M}{2} \sum_{i=1,2} \left( \frac{\partial y_i}{\partial q_i} \right)^T \dot{q}_i \, dt \\
&= \dot{M}(0) + \left( \gamma M \right) \frac{1}{2} \sum_{i=1,2} \left( \frac{\partial y_i}{\partial q_i} \right)^T \dot{q}_i.
\end{align*}
\]

and \( \gamma_M, \gamma_{Niz}(i = 1, 2), \gamma_{Nz2}, \) and \( \gamma_{Nz2} \) are positive constants. The form is nothing but the control signal proposed in the rigid contact case \([11] [13]\). The third term of the constants. The form is nothing but the control signal proposed

\[
\begin{align*}
\dot{N}_{\Delta x_i} &= \gamma_{Niz} - \frac{1}{N} \int_0^t (r_1 e_{\Delta x_i}) \, dt \\
&= \frac{1}{N} \int_0^t (r_2 e_{\Delta x_i}) \, dt.
\end{align*}
\]

Differently from the case of rigid finger-ends \([11]\), \( f_0 \) is not a constant but dependent on the magnitude of \( \Delta x_1 + \Delta x_2 \). Nevertheless, it is possible to find \( \Delta x_{di}(i = 1, 2) \) for a given \( f_d > 0 \) so that they satisfy

\[
\tilde{f}_i(\Delta x_{di}) = \left( 1 + \frac{l_1 + l_2 - \Delta x_1 - \Delta x_2}{r_1 + r_2} \right) f_d, \quad i = 1, 2
\]

because \( \tilde{f}_i(\Delta x) \) is of the form of \( \tilde{f}_i(\Delta x) = k_i \Delta x^2 \) (Equation (21)). Substituting this control signals (Equation (39)) into Equations (33), (34), and (35) yields

\[
\begin{align*}
H_i \dot{q}_i &= \left( \frac{1}{2} H_i + H_i + \hat{C}_i \right) \dot{q}_i + \frac{\partial R_i}{\partial q_i} - \left( -1 \right)^i J_{0i}^T \dot{f}_i \dot{r}_x \\
- \Delta \lambda \dot{Y} q_1 - \Delta \lambda \dot{Z}_1 q_1 + M g (\frac{\partial y_i}{\partial q_i} + r_i \Delta N_{i} e_{\Delta x_i} + r_i \Delta N_{i} e_{\Delta x_i} + g) = 0, \quad i = 1, 2 \\
M \ddot{\Delta x} - (f_1 - f_2) \dot{r}_x + (\Delta \lambda \dot{Y} + \Delta \lambda \dot{Z}_2) \dot{r}_y + (\Delta \lambda \dot{Z}_1 + \Delta \lambda \dot{Z}_2) \dot{r}_z = 0
\end{align*}
\]
where
\[
\Delta f_i = f_i - f_0 - (-1)^i \frac{Mg}{2} r_{xy} \tag{49}
\]
\[
\Delta \lambda_{yi} = \lambda_{yi} + (-1)^i \frac{f_d}{r_1 + r_2} (Y_i - Y_2) - \frac{Mg}{2} r_{yy} \tag{50}
\]
\[
\Delta z_i = z_i + (-1)^i \frac{f_d}{r_1 + r_2} (Z_i - Z_2) - \frac{Mg}{2} r_{zy} \tag{51}
\]
\[
\Delta M = M - M, \quad \Delta N_i = N_i - N_i, \quad i = 1, 2
\]
\[
\Delta N_{2x} = N_{2x} - N_{2x}, \quad \Delta N_{22} = N_{22} - N_{22} \tag{52}
\]
\[
N_{iz} = \frac{f_d}{r_1 + r_2} (r_i - \Delta x_i) \left\{ (Y_i - Y_2) r_{yz} - (Z_i - Z_2) r_{yz} - (-1)^i (r_i - \Delta x_i) \frac{Mg}{2} r_{yy} r_{yz} - r_{zy} r_{xy} \right\}, \quad i = 1, 2 \tag{53}
\]
\[
N_{2x} = \frac{f_d}{r_1 + r_2} (r_2 - \Delta x_2) \left\{ (Y_1 - Y_2) r_{yz} - (Z_1 - Z_2) r_{xy} \right\} - \frac{r_2}{r_2} (r_2 - \Delta x_2) \frac{Mg}{2} r_{yy} r_{yz} - r_{zy} r_{xy} \tag{54}
\]
\[
N_{22} = \frac{f_d}{r_1 + r_2} (r_2 - \Delta x_2) \left\{ (Y_1 - Y_2) r_{Z21} r_{yz} - (Z_1 - Z_2) r_{Z21} r_{xy} \right\} - \frac{r_2}{r_2} (r_2 - \Delta x_2) \frac{Mg}{2} r_{Z21} r_{yy} r_{yz} - r_{Z21} r_{xy} r_{zy} \tag{55}
\]
\[
S_x = \frac{Mg}{2} \left\{ (Z_1 + Z_2) r_{yy} - (Y_1 + Y_2) r_{yz} \right\} \tag{56}
\]
\[
S_y = \frac{Mg}{2} \left\{ (Z_1 + Z_2) r_{xy} + r_{zy} (l_1 - l_2) \right\} \tag{57}
\]
\[
S_z = \frac{Mg}{2} \left\{ (Z_1 + Z_2) r_{xy} + r_{zy} (l_1 - l_2) \right\} \tag{58}
\]

V. NUMERICAL SIMULATION RESULTS AND INITIAL VALUES

We carry out numerical simulations based on the physical parameters of the fingers-object system given in Table II and the parameters of control gains given in Table III in order to confirm the stability of motion of the overall dynamics applied by the proposed control input. It is noted that \(\delta q_{21x}\) and \(\delta q_{21z}\) have a meaning as a virtual displacement but quantity of \(\delta q_{21x}\) and \(\delta q_{21z}\) do not have physical meanings of rotational angles. Hence we set the virtual angles \(q_{21x}(0)\) and \(q_{21z}(0)\) to zero respectively as shown Table III and the virtual angles do not directly affect numerical simulations at all. The results of numerical simulations, representing key variables of fingers-object system, converge to some constant values as shown in Fig 7. It is confirmed that spinning motion arises but eventually it ceases after 10 seconds, that is, \(\omega \to 0\) as \(t \to \infty\) seen through Figs.6 and 7. On the other hand, if we set the damping parameters \(c_{11}\) and \(c_{12}\) to zero respectively, we can see that spinning motion continues eternally. Therefore it is confirmed that motion of the overall fingers-object system is stabilized from the viewpoint of simulations.

![Fig. 6. Motions of pinching a 3-D object under the gravity effect](image-url)
VI. CONCLUSION

A saddle joint model with orderless actuations and rolling constraints have been treated in a faithful manner and incorporated into the overall fingers-object system. It is shown that 3-D pinching by two robot fingers, one is a 3-DOFs planar finger (resembles the index finger) and the other is a 3-DOFs finger with a saddle joint (the thumb), is performed numerically by using the derived computational model. Differently from the previous paper [10], the overall fingers-object dynamics are derived as a full-variables model. It is shown through computer simulations that spinning motion between finger tips and the object arises but eventually stops by virtue of the incorporated viscosity model exerted on rotational motion of the object around $r_X$ vector at two contact points between a left or right finger tip and the object surface. Numerical simulation results showed the stability of motion of the overall fingers-object dynamics to which the proposed control input is applied. However, to confirm the effectiveness of our proposed control input in a sound meaning, the analysis of its stability is indispensable and should be carried out from the mathematically reasonable viewpoint. Finally it is shown that the proposed viscosity model plays a crucial role in pinching a 3-D object.

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